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On the integer solutions of the Pell equation $x^2 - 18y^2 = 4^k$

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ABSTRACT*: The binary quadratic diophantine equation represented by $x^2 - 18y^2 = 4^k, k > 0$ is considered. A method for obtaining infinitely many non-zero distinct integer solutions of the Pell equation considered above is illustrated. A few interesting relations among the solutions and special figurate numbers are presented.Recurrence relations on the solutions are given.*

KEYWORDS - *Pell equation, binary quadratic diophantine equation, integer solutions.*

I. INTRODUCTION

It is well known that the Pell equation $x^2 - Dy^2 = 1$ (D > 0 and square free) has always positive integer solutions.When $N \neq 1$, the Pell equation $x^2 - Dy^2 = N$ may not have any positive integer solutions.For example, the equations $x^2 = 3y^2 - 1$ and $x^2 = 7y^2 - 4$ have no integer solutions. When k is a positive integer and $D \in (k^2 \pm 4, k^2 \pm 1)$, positive integer solutions of the equations $x^2 - Dy^2 = \pm 4$ and $x^2 - Dy^2 = \pm 1$ have been investigated by Jones in [1].In [2-11], some specific Pell equation and their integer solutions are considered.In [12], the integer solutions of the Pell equation $x^2 - (k^2 + k)y^2 = 2^t$ has been considered. In [13], the Pell equation $x^2 - (k^2 - k)y^2 = 2^t$ is analysed for the integer solutions.

This communication concerns with the Pell equation $x^2 - 18y^2 = 4^k, k > 0$ and infinitely many positive integer solutions are obtained.A few interesting relations among the solutions and special figurate numbers are presented.Recurrence relations on the solutions are given.

II. Notations

$t_{m,n}$ - Polygonal number of rank n with sides m

P_n^m - Pyramidal number of rank n with sides m

CP_n^m - Centered Pyramidal number of rank n with sides m

PCS_m^n - Prism number of rank n with sides m

$G(n)$- Gnomonic number

$SO(n)$- Stella octangula number

$CD(n)$- Centered Dodecahedral number

$CC(n)$- Centered Cube number

$TOH(n)$-Truncated Octahedral number

$PTP(n)$-Pentatope number

$HO(n)$- Hauy Octahedral number

$N_d(n)$- n^{th} d-dimensional nexus number

$g_m(n)$- m-gram number of rank n

$RD(n)$- Rhombic Dodecahedral number

Method of ANALYSIS

The Pell equation to be solved is $x^2 - 18y^2 = 4^k$ (1)

Let (X_0, Y_0) be the initial solution of (1) which is given by

$$X_0 = 17.2^k \ ; Y_0 = 2^{k+2} \ , \ k \in z - \{0\}$$

To find the other solutions of (1), consider the Pellian equation

$$x^2 = 18y^2 + 1$$

whose general solution $(\tilde{x}_n, \tilde{y}_n)$ is given by

$$\tilde{x}_n = \frac{1}{2} f_n$$

$$\tilde{y}_n = \frac{1}{6\sqrt{2}} g_n$$

where $f_n = \left(17 + 12\sqrt{2}\right)^{n+1} + \left(17 - 12\sqrt{2}\right)^{n+1}$

$$g_n = \left(17 + 12\sqrt{2}\right)^{n+1} - \left(17 - 12\sqrt{2}\right)^{n+1} \ , \ n = 0,1,2, \ldots$$

Applying Brahmagupta lemma between (X_0, Y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of non-zero distinct integer to (1) are obtained as

$$X_{n+1} = 2^{k-1}(17 f_n + 12\sqrt{2} g_n) \quad (2)$$

$$Y_{n+1} = \frac{2^{k-1}}{3\sqrt{2}}(12\sqrt{2} f_n + 17 g_n) \quad (3)$$

The recurrence relations satisfied by the solutions of (1) are given by

$$X_{n+3} - 34X_{n+2} + X_{n+1} = 0 \ , \quad X_1 = 577.2^k, X_2 = 19601.2^k$$

$$Y_{n+3} - 34Y_{n+2} + Y_{n+1} = 0 \ , \quad Y_1 = 17.2^{k+3}, Y_2 = 1155.2^{k+2}$$

From (2) and (3), the values of f_n and g_n are found to be

$$f_n = \frac{1}{2^k}(34X_{n+1} - 144Y_{n+1}) \ ; \ g_n = \frac{1}{2^k\sqrt{2}}(204Y_{n+1} - 48X_{n+1}) \quad (4)$$

In view of (4), the following relations are observed

1. $24(34X_{n+1} - 144Y_{n+1})^2 - 12(204Y_{n+1} - 48X_{n+1})^2$ is a nasty number.
2. $X_{n+2} = 17X_{n+1} + 72y_{n+1}$
3. $X_{n+3} = 577X_{n+1} + 2448y_{n+1}$
4. $Y_{n+2} = 4X_{n+1} + 17Y_{n+1}$
5. $Y_{n+3} = 136X_{n+1} + 577Y_{n+1}$
6. $Y_{n+1} = Y_{n+3} - 8X_{n+2}$
7. $Y_{n+1} = 17Y_{n+2} - 4X_{n+2}$
8. $34X_{2n+2} - 144Y_{2n+2} + 2^{k+1} \equiv 0 (mod\ 2^k)$
9. $17X_{2n+2} - 72Y_{2n+2} - g_3(f_n).2^{k-1} + t_{6,f_n}.2^{k-1} + G(f_n).2^{k-1} \equiv 0(mod\ 2^k)$

10. $34X_{3n+3} - 144Y_{3n+3} = \left(CP_{f_n}^{9} - 2CP_{f_n}^{3}\right).2^{k+1}$
11. When $k \equiv 0(mod\ 3)$, $34X_{3n+3} - 144Y_{3n+3} + 3(34X_{n+1} - 144Y_{n+1})$ is a cubic integer.
12. $34X_{3n+3} - 144Y_{3n+3} - P_{f_n}^{5}.2^{k+1} + 3t_{3,f_n}.2^{k+1} \equiv 0(mod\ f_n)$
13. $34X_{3n+3} - 144Y_{3n+3} - PCS_4^{f_n}.2^{k+1} + 3HO(f_n)2^k + t_{12,f_n}2^k - 3P_{f_n}^{3}2^{k+1} \equiv 0(mod\ 3)$
14. $17X_{3n+3} - 72Y_{3n+3} - SO(f_n).2^{k-1} + CP_{f_n}^{3}.2^k \equiv 0(mod\ f_n)$
15. $17X_{3n+3} - 72Y_{3n+3} - P_{f_n}^{3}.2^k + G(f_n).2^k \equiv 0(mod\ 2^k)$
16. $34X_{3n+3} - 144Y_{3n+3} - TOH(f_n)2^k + 3CP_{f_n}^{15}.2^{k+1} - 3t_{18,f_n}.2^k \equiv 0(mod\ 6)$
17. $5(34X_{4n+4} - 144Y_{4n+4}) = 5.2^{k+1} + N_4(f_n)2^k - CD(f_n)2^k - 13t_{4,f_n}2^k - t_{3,f_n-1}.2^{k+2}$
18. $34X_{4n+4} - 144Y_{4n+4} = 3PTP(f_n).2^{k+3} - CC(f_n).2^k + RD(f_n).2^k + 11t_{4,f_n}.2^k + 3t_{3,f_n}.2^{k+1}$
19. When $k \equiv 0(mod\ 4)$, $34X_{4n+4} - 144Y_{4n+4} + t_{4,f_n}.2^{k+2} - 2^{k+1}$ is a biquadratic integer.
20. Define $X = 34X_{n+1} - 144Y_{n+1}$ and $Y = 204Y_{n+1} - 48X_{n+1}$. Note that (X, Y) satisfies the hyperbola $Y^2 = 2X^2 - 32$.

III. CONCLUSION

To conclude, one may search for other patterns of solutions to the similar equation considered above.

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